Solutions to Question Sheet 6, Differentiation I. v1 2019-20

Verifying the Definition

In these questions we use the Limit Laws to verifying the definition

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}. \tag{1}$$

We will also look at examples where the limit fails to exist so f is not differentiable at a, and where we have to look at both one-sided limits to show that the limit in (1) exists, or not.

1. Using the *definition* of the derivative as a limit, and **not** the differentiation rules, calculate the derivatives of the following functions.

   i) $x^4, \quad x \in \mathbb{R}$ ii) $\sqrt{x}, \ \ x > 0$ iii) $\dfrac{1}{1+x^4}, \ \ x \in \mathbb{R}$.

   Solution i) Let $a \in \mathbb{R}$ be given. Observe that for $x \neq a$,

   $$\frac{x^4-a^4}{x-a} = \frac{(x-a)\left(x^3+ax^2+a^2x+a^3\right)}{x-a} = x^3+ax^2+a^2x+a^3.$$

   So
   $$\lim_{x\to a}\frac{x^4-a^4}{x-a} = \lim_{x\to a}\left(x^3+ax^2+a^2x+a^3\right).$$

   But polynomials are continuous for all x, so the limit at a is simply the value of the polynomial at a, in this case $4a^3$. Since the limit exists the function x^4 is differentiable at a, with derivative $4a^3$.

   Yet $a \in \mathbb{R}$ was arbitrary so x^4 is differentiable on $\mathbb{R}$ with

   $$\frac{d}{dx}x^4 = 4x^3.$$

   ii) Let $a > 0$ be given. Observe that for $x \neq a$ and $x > 0$,

   $$\frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)} = \frac{1}{\sqrt{x}+\sqrt{a}}.$$

So

$$\begin{aligned}
\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} &= \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} \\
&= \frac{1}{\lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right)} \\
&\qquad\qquad \text{by the Quotient Rule for Limits,} \\
&= \frac{1}{2\sqrt{a}}. \qquad (2)
\end{aligned}$$

Here we have used the result seen in Question 3ii, Sheet 4, that $\sqrt{x}$ is *continuous* for $x > 0$ and so the limit at $a > 0$ equals the value at a. For the Quotient Rule we have used the fact that $\lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right) = \sqrt{a} \neq 0$. Since the limit exists the function $\sqrt{x}$ is differentiable at a, with derivative $1/\left(2\sqrt{a}\right)$.

Yet $a \in \mathbb{R}^+$ was arbitrary so $\sqrt{x}$ is differentiable on $\mathbb{R}^+$ with

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

iii) Let $a \in \mathbb{R}$ be given. Observe that for $x \neq a$,

$$\begin{aligned}
\frac{\frac{1}{1+x^4}-\frac{1}{1+a^4}}{x-a} &= \frac{(1+a^4)-(1+x^4)}{(1+a^4)(1+x^4)(x-a)} \\
&= -\frac{x^4-a^4}{(1+a^4)(1+x^4)(x-a)} \\
&= -\frac{x^3+ax^2+a^2x+a^3}{(1+a^4)(1+x^4)},
\end{aligned}$$

using the ideas seen in Part i. So, by the Quotient Rule for Limits,

$$\lim_{x\to a}\frac{\frac{1}{1+x^4}-\frac{1}{1+a^4}}{x-a} = -\frac{1}{(1+a^4)}\frac{\lim_{x\to a}\left(x^3+ax^2+a^2x+a^3\right)}{\lim_{x\to a}\left(1+x^4\right)} = -\frac{4a^3}{\left(1+a^4\right)^2}.$$

In the final equality we have used the fact that a polynomial is continuous and so the limit at a equals the value of the polynomial at a. Since the limit exists the function $1/(1+x^4)$ is differentiable at a, with derivative $-4a^3/\left(1+a^4\right)^2$.

Yet $a \in \mathbb{R}$ was arbitrary so $1/(1+x^4)$ is differentiable on $\mathbb{R}$ with

$$\frac{d}{dx}\frac{1}{1+x^4} = -\frac{4x^3}{(1+x^4)^2}.$$

2. Recall the results from the Lecture Notes that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Assume the addition formulae for cosine and tangent.

Prove, by verifying the *definition* that,

i)
$$\frac{d}{dx}\cos = -\sin x,$$
for $x \in \mathbb{R}$,

ii)
$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$
for $x \notin \left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$.

Solution i) Let $a \in \mathbb{R}$ be given. Then

$$\begin{aligned}
\lim_{y\to 0}\frac{\cos(y+a) - \cos a}{y} &= \lim_{y\to 0}\frac{\cos y\cos a - \sin y \sin a - \cos a}{y} \\
&= \cos a \lim_{y\to 0}\left(\frac{\cos y - 1}{y}\right) - \sin a \lim_{y\to 0}\frac{\sin y}{y} \\
&\qquad \text{by the Sum Rule for Limits} \\
&= \cos a \times 0 - 1 \times \sin a \\
&\qquad \text{by the recollection in the question} \\
&= -\sin a,
\end{aligned}$$

Since the limit exists $\cos x$ is differentiable at a with derivative $-\sin a$.

Yet $a \in \mathbb{R}$ was arbitrary so $\cos x$ is differentiable on $\mathbb{R}$ with

$$\frac{d}{dx}\cos x = -\sin x.$$

ii) Let $a \in \mathbb{R}$, but not of the form $\pi/2 + n\pi$ for any $n \in \mathbb{Z}$, be given. Then

$$\lim_{y\to 0}\frac{\tan(a+y)-\tan a}{y} = \lim_{y\to 0}\frac{1}{y}\left(\frac{\tan a+\tan y}{1-\tan a\tan y}-\tan a\right),$$

by the sum formula for the tangent. This equals

$$\lim_{y\to 0}\frac{\tan y}{y}\left(\frac{1+\tan^2 a}{1-\tan a\tan y}\right) = \lim_{y\to 0}\frac{\sin y}{y}\left(\frac{1}{\cos y-\tan a\sin y}\right)\frac{1}{\cos^2 a},$$

having used

$$1+\tan^2 a = \frac{\cos^2 a+\sin^2 a}{\cos^2 a} = \frac{1}{\cos^2 a}.$$

By the Quotient and Sum Rules for *limits* this equals

$$\frac{1}{\cos^2 a}\left(\lim_{y\to 0}\frac{\sin y}{y}\right)\frac{1}{\lim_{y\to 0}\cos y-\tan a\lim_{y\to 0}\sin y}$$

$$= \frac{1}{\cos^2 a}\times 1\times\frac{1}{1-0\times\tan a} = \frac{1}{\cos^2 a}.$$

Since the limit exists $\tan x$ is differentiable at a with derivative $1/\cos^2 a$. Yet $a \in \mathbb{R}$ was arbitrary subject to $a \neq \pi/2 + n\pi$ for any $n \in \mathbb{Z}$, so $\tan x$ is differentiable for all real $x \neq \pi/2 + n\pi$ for any $n \in \mathbb{Z}$ and

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}.$$

Aside You were asked to verify the definition, but if you had not been so restricted you might have applied the *Quotient Rule* for differentiation. Then

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\frac{d\sin x}{dx}-\sin x\frac{d\cos x}{dx}}{\cos^2 x},$$

provided $\cos^2 x \neq 0$, i.e. $x \neq \pi/2+n\pi$ for any $n \in \mathbb{Z}$. Using the results proved for $\sin x$ and $\cos x$ we have

$$\frac{d}{dx}\tan x = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

valid for $x : \cos x \neq 0$ i.e. $x \notin \{(1+2n)\,\pi/2 : n \in \mathbb{Z}\}$.

3. Recall the result from the Notes that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Use this, and the *definition* of derivative, to find the derivatives of

i) e^{2x} ii) xe^x. iii) $\sinh x$.

Solution i) **I give two solutions. First,** let $a \in \mathbb{R}$ be given. Observe that for $h \neq 0$,

$$\begin{aligned}
\frac{f(a+h) - f(a)}{h} &= \frac{e^{2(a+h)} - e^{2a}}{h} \quad \text{having written } x = a+h, \\
&= e^{2a} \left(\frac{e^{2h} - 1}{h} \right) \quad \text{having used } e^{a+h} = e^a e^h, \\
&= e^{2a} \left(\frac{e^h - 1}{h} \right) \left(e^h + 1 \right),
\end{aligned}$$

since $e^{2h} - 1 = \left(e^h - 1\right)\left(e^h + 1\right)$. Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} e^{2a} \left(\frac{e^h - 1}{h} \right) \left(e^h + 1 \right) \\
&= e^{2a} \lim_{h\to 0} \left(\frac{e^h - 1}{h} \right) \times \lim_{h\to 0} \left(e^h + 1 \right) \\
&\qquad \text{by Product Rule for Limits,} \\
&= 2e^{2a}.
\end{aligned}$$

Since the limit exists the function e^{2x} is differentiable at a, with derivative $2e^{2a}$.

Yet $a \in \mathbb{R}$ was arbitrary so e^{2x} is differentiable on $\mathbb{R}$ with

$$\frac{de^{2x}}{dx} = 2e^{2x}.$$

Second proof, let $a \in \mathbb{R}$ be given. Observe that for $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{e^{2x} - e^{2a}}{x - a} = \frac{\left(e^x - e^a\right)\left(e^x + e^a\right)}{x - a}.$$

Take the limit $x \to a$, using the Product Rule for limits, allowable since both individual limits exist since we know that e^x is differentiable on $\mathbb{R}$. Thus

$$\begin{aligned} \lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} \frac{(e^x-e^a)}{x-a} \lim_{x\to a} (e^x+e^a) \\ &= e^a \times 2e^a = 2e^{2a.} \end{aligned}$$

Since the limit exists the function e^{2x} is differentiable at a, with derivative $2e^{2a}$.

Yet $a \in \mathbb{R}$ was arbitrary so e^{2x} is differentiable on $\mathbb{R}$ with

$$\frac{de^{2x}}{dx} = 2e^{2x}.$$

ii) **I give two solutions. First,** let $a \in \mathbb{R}$ be given. Observe that for $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)\, e^{a+h} - ae^a}{h} = ae^a \frac{e^h-1}{h} + e^a e^h.$$

Now use the Sum Rule for Limits to get

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = ae^a \lim_{h\to 0} \frac{e^h-1}{h} + e^a \lim_{h\to 0} e^h = ae^a + e^a.$$

Since the limit exists the function xe^x is differentiable at a, with derivative $ae^a + e^a$.

Yet $a \in \mathbb{R}$ was arbitrary so xe^x is differentiable on $\mathbb{R}$ with

$$\frac{dxe^x}{dx} = xe^x + e^x.$$

Second Solution. Let $a \in \mathbb{R}$ be given. Observe that for $x \neq a$,

$$\frac{f(x)-f(a)}{x-a} = \frac{xe^x - ae^a}{x-a} = \frac{xe^x - ae^x + ae^x - ae^a}{x-a},$$

having added in zero, $0 = -ae^x + ae^x$. Continuing, this equals

$$\frac{xe-a}{x-a}e^x + a\frac{e^x-e^a}{x-a} = e^x + a\frac{e^x-e^a}{x-a}.$$

Thus, by the Sum Rule for Limits,

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a} e^x + a\lim_{x\to a}\frac{e^x-e^a}{x-a} = e^a + ae^a,$$

the second limit following from the fact that e^x is differentiable on $\mathbb{R}$.

Yet $a \in \mathbb{R}$ was arbitrary so xe^x is differentiable on $\mathbb{R}$ with

$$\frac{dxe^x}{dx} = xe^x + e^x.$$

iii) Recall the definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

for $x \in \mathbb{R}$. Let $a \in \mathbb{R}$ be given. With $h \neq 0$ we have

$$\begin{aligned}
\frac{\sinh(a+h) - \sinh a}{h} &= \frac{1}{h}\left(\frac{e^{a+h}-e^{-a-h}}{2} - \frac{e^a - e^{-a}}{2}\right) \\
&= \frac{1}{2h}\left(\left(e^{a+h} - e^a\right) - \left(e^{-a-h} - e^{-a}\right)\right) \\
&= \frac{e^a}{2}\left(\frac{e^h - 1}{h}\right) - \frac{e^{-a}e^{-h}}{2}\left(\frac{1-e^h}{h}\right) \\
&= \frac{1}{2}\left(e^a + \frac{e^{-a}}{e^h}\right)\left(\frac{e^h-1}{h}\right).
\end{aligned}$$

Now use the Rules for Limits to get

$$\begin{aligned}\lim_{h\to 0}\frac{\sinh(a+h)-\sinh a}{h} &= \frac{1}{2}\left(e^a+\frac{e^{-a}}{\lim_{h\to 0}e^h}\right)\lim_{h\to 0}\left(\frac{e^h-1}{h}\right),\\ &= \frac{1}{2}\left(e^a+e^{-a}\right)=\cosh a,\end{aligned}$$

allowable since both limits exist and $\lim_{h\to 0} e^h = 1 \neq 0$. Therefore

$$\lim_{h\to 0}\frac{\sinh(a+h)-\sinh a}{h}=\frac{1}{2}\left(e^a+e^{-a}\right)=\cosh a.$$

Since the limit exists the function $\sinh x$ is differentiable at a, with derivative $\cosh a$.

Yet $a \in \mathbb{R}$ was arbitrary so $\sinh x$ is differentiable on $\mathbb{R}$ with

$$\frac{d\sinh x}{dx}=\cosh x.$$

4. Use the *definition* of derivative to find

$$\frac{d}{dx}\left(e^x\sin x\right)$$

for $x \in \mathbb{R}$.

(You may assume if necessary, that $\sin(a+h)=\sin a\cos h+\cos a\sin h$).

Hint Do not use the result but look at the *proof* of the Product Rule for differentiation and use the idea of "adding in zero".

Solution I give two solutions. First, let $a \in \mathbb{R}$ be given and consider, for $h \neq 0$,

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{e^{a+h}\sin(a+h)-e^a\sin a}{h}\\ &= e^a\frac{e^h\left(\sin a\cos h+\cos a\sin h\right)-\sin a}{h}\\ &= e^a\frac{e^h\cos h-1}{h}\sin a+e^a\frac{e^h\sin h}{h}\cos a.\end{aligned}$$

Now use the hint given in the question and "add in zero" in the form $0 = -\cos h + \cos h$. Then

$$\begin{aligned}\frac{e^h \cos h - 1}{h} &= \frac{\left(e^h - 1\right)\cos h + \cos h - 1}{h} \\ &= \cos h \frac{e^h - 1}{h} + \frac{\cos h - 1}{h.}\end{aligned}$$

Use the Sum and Product Rules for limits to get

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= e^a \sin a \lim_{h\to 0} \frac{\left(e^h - 1\right)}{h} \lim_{h\to 0} \cos h + e^a \sin a \lim_{h\to 0} \frac{\cos h - 1}{h} \\ &\qquad + e^a \cos a \lim_{h\to 0} e^h \lim_{h\to 0} \frac{\sin h}{h}. \\ &= e^a \sin a \times 1 \times 1 \ + \ e^a \sin a \times 0 \ + \ e^a \cos a \times 1 \times 1 \\ &= e^a \sin a + e^a \cos a.\end{aligned}$$

Since the limit exists the function $e^x \sin x$ is differentiable at a, with derivative $e^a \sin a + e^a \cos a$.

Yet $a \in \mathbb{R}$ was arbitrary so $e^x \sin x$ is differentiable on $\mathbb{R}$ with

$$\frac{de^x \sin x}{dx} = e^x \sin x + e^x \cos x.$$

Second Solution. Let $a > 0$ be given. Consider

$$\begin{aligned}\frac{f(x) - f(a)}{x - a} &= \frac{e^x \sin x - e^a \sin a}{x - a} \\ &= \frac{e^x \sin x - e^a \sin x + e^a \sin x - e^a \sin a}{x - a}, \qquad (3)\end{aligned}$$

having again added in zero, this time of the form $0 = -e^a \sin x + e^a \sin x$. Continuing, (3) equals

$$\sin x \frac{e^x - e^a}{x - a} + e^a \frac{\sin x - \sin a}{x - a}.$$

Take the limit as $x \to a$ and use the Product and Sum Rules for limits. This is allowable sine all the individual limits exist because we know that e^x and $\sin x$ are differentiable on $\mathbb{R}$. Thus,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\sin x \lim_{x\to a}\frac{e^x-e^a}{x-a}+\lim_{x\to a}e^a\lim_{x\to a}\frac{\sin x-\sin a}{x-a}\\ &= \sin a\times e^a+e^a\times\cos a.\end{aligned}$$

Since the limit exists the function $e^x\sin x$ is differentiable at a, with derivative $e^a\sin a+e^a\cos a$.

Yet $a\in\mathbb{R}$ was arbitrary so $e^x\sin x$ is differentiable on $\mathbb{R}$ with

$$\frac{de^x\sin x}{dx}=e^x\sin x+e^x\cos x$$

5. i) Prove that $|\sin\theta|$ is **not** differentiable at $\theta=0$.

   ii) Prove, by verifying the definition, that $|\sin\theta|\sin\theta$ **is** differentiable at $\theta=0$, and find the value of the derivative.

   You may assume that $\lim_{\theta\to 0}\sin\theta=0$ and $\lim_{\theta\to 0}(\sin\theta)/\theta=1$.

Solution i. If $0<\theta<\pi/2$ then $\sin\theta>0$ so $|\sin\theta|=\sin\theta$. Thus

$$\frac{|\sin\theta|-|\sin 0|}{\theta-0}=\frac{\sin\theta}{\theta}\to 1$$

as $\theta\to 0+$ by assumption in question.

If $-\pi/2<\theta<0$ then $\sin\theta<0$ and $|\sin\theta|=-\sin\theta$. Thus

$$\frac{|\sin\theta|-|\sin 0|}{\theta-0}=\frac{-\sin\theta}{\theta}\to -1$$

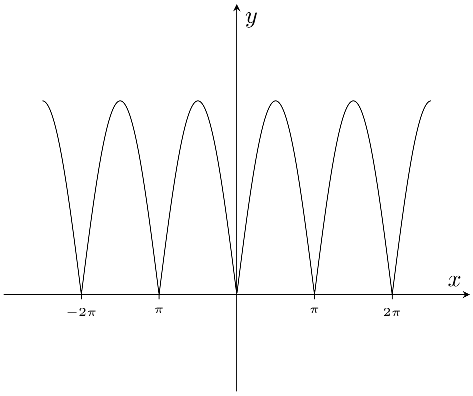
as $\theta\to 0-$ by assumption in question.

Since the one-sided limits are different we conclude that

$$\lim_{\theta\to 0}\frac{|\sin\theta|-|\sin 0|}{\theta-0}$$

does not exist and hence $|\sin\theta|$ is **not** differentiable at $\theta=0$.

The graph of $|\sin\theta|$ is

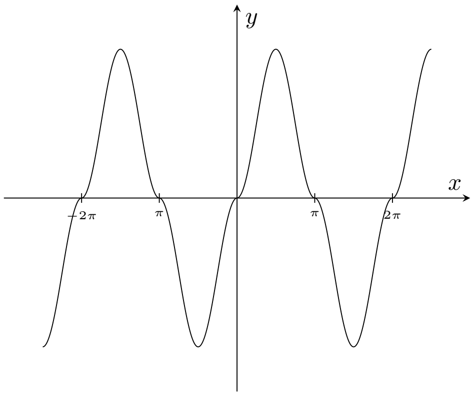


ii. For $\theta \neq 0$ consider

$$\frac{|\sin\theta|\sin\theta - |\sin 0|\sin 0}{\theta - 0} = |\theta|\left|\frac{\sin\theta}{\theta}\right|\left(\frac{\sin\theta}{\theta}\right) \to 0$$

as $\theta \to 0$ by the Product Rule for limits and the assumption of the question. Because the limit exists $|\sin\theta|\sin\theta$ is differentiable at $\theta = 0$, and the value of the derivative is 0.

The graph of $|\sin\theta|\sin\theta$ is



6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^2 + 4x - 12}{x^2 - 4} & \text{if } x \neq 2, -2 \\ 2 & \text{if } x = 2 \\ 1 & \text{if } x = -2. \end{cases}$$

i) Prove, by verifying the definition, that $f(x)$ is differentiable at $x = 2$, and find the value of the derivative.

ii) Prove that $f(x)$ is not differentiable at $x = -2$.

Solution i) For $x \neq 2$ or -2 consider

$$\begin{aligned} \frac{f(x) - f(2)}{x - 2} &= \frac{1}{(x-2)}\left(\frac{x^2 + 4x - 12}{x^2 - 4} - 2\right) \\ &= \frac{1}{(x-2)} \frac{-x^2 + 4x - 4}{(x-2)(x+2)} \\ &= -\frac{(x-2)^2}{(x-2)^2 (x+2)} = -\frac{1}{x+2} \\ &\to -\frac{1}{4} \end{aligned}$$

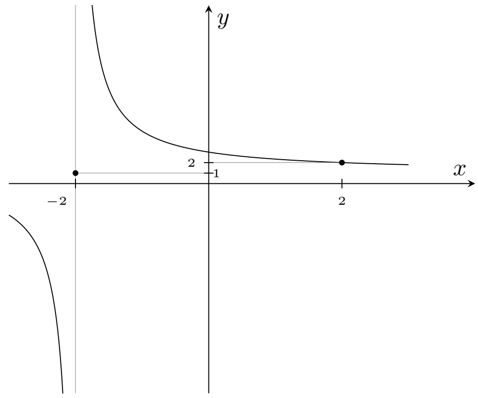
as $x \to 2$. Since the limit exists $f(x)$ **is** differentiable at $x = 2$, with derivative $-1/4$.

ii. For $x \neq 2$ and -2 consider

$$\frac{f(x) - f(-2)}{x - (-2)} = \frac{1}{(x+2)}\left(\frac{x^2 + 4x - 12}{x^2 - 4} - 1\right) = \frac{4}{(x+2)^2}.$$

This does not have a finite limit as $x \to -2$ and so f is **not** differentiable at $x = -2$.

The graph of f is



7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 + 1 & \text{if } x < 1 \end{cases}.$$

By verifying the *definition* prove that f is differentiable at $x = 1$ and find the value of the derivative.

Solution A function f is differentiable at a iff $\lim_{x \to a} (f(x) - f(a))/(x - a)$ exists and for this it suffices to show that both one-sided limits exist and are equal.

For this question, if $x < 1$, we have $f(x) = x^2 + 1$ in which case

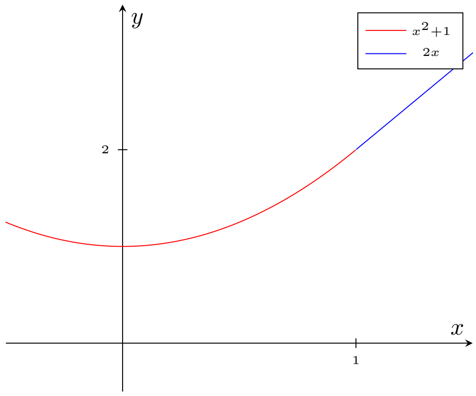
$$\begin{aligned} \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \to 1-} \frac{(x^2 + 1) - 2}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1-} (x + 1) = 2. \end{aligned}$$

If $x > 1$, then $f(x) = 2x$ in which case

$$\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{2x - 2}{x - 1} = \lim_{x \to 1+} 2 = 2.$$

The one-sided limits exist and are equal, so f is differentiable at $x = 1$. The common value, 2, is the value of the derivative there, i.e. $f'(1) = 2$.

The graph of f is



8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - x & \text{for } x \leq 1 \\ x^3 - 1 & \text{for } x > 1. \end{cases}$$

Prove that f is **not** differentiable at $x = 1$.

(It is quickly seen that the one-sided limits of f at $x = 1$ are both 0, the value of $f(0)$, and so f is continuous at $x = 1$. Thus we have another example that continuous does not imply differentiable.)

Solution By definition $f(1) = 0$. If $x \leq 1$, then $f(x) = x^2 - x$ and

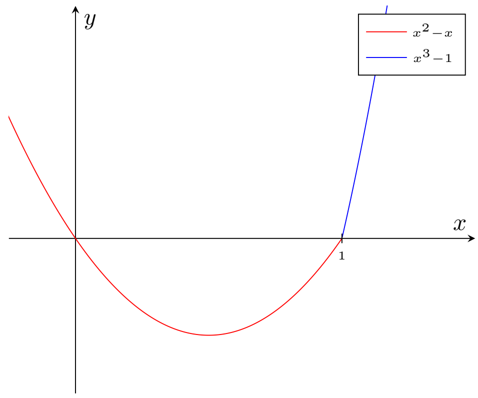
$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x^2 - x - 0}{x - 1} = \lim_{x \to 1-} x = 1.$$

If $x > 1$ then $f(x) = x^3 - 1$ so

$$\begin{aligned} \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \to 1+} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1+} \frac{(x-1)\left(x^2 + x + 1\right)}{x - 1} \\ &= \lim_{x \to 1+} \left(x^2 + x + 1\right) = 3. \end{aligned}$$

Since the two one-sided limits are different the limit does not exist and so f is **not** differentiable at $x = 1$.

The graph of f is



9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

   i) Use the *definition* to show that f is differentiable at $x = 0$ and find the value of $f'(0)$.

   ii) Find $f'(x)$ for **all** $x \in \mathbb{R}$.

   iii) Is the derivative f' differentiable on $\mathbb{R}$? Give your reasons.

Solution i) By definition $f(0) = 0$. Consider first $x \geq 0$ when $f(x) = x^2$ and

$$\lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{x^2}{x} = \lim_{x\to 0+} x = 0.$$

Next, when $x < 0$ we have $f(x) = -x^2$ so

$$\lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{-x^2}{x} = -\lim_{x\to 0+} x = 0.$$

Hence, because both one-sided limits exist and are equal, the limit as $x \to 0$ exists and

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

ii) For $x \neq 0$, then $f(x)$ equals either x^2 or $-x^2$ when $f'(x) = 2x$ or $-2x$. Hence

$$f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -2x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0, \\ -2x & \text{if } x < 0. \end{cases}$$

iii) We next try to differentiate f' at $x = 0$. Look at the two one-sided limits:

$$\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{2x}{x} = 2,$$

while

$$\lim_{x \to 0-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0-} \frac{-2x}{x} = -2.$$

Since the two-one sided limits are different, the limit as $x \to 0$ does not exist, i.e. f' does **not** have a derivative at $x = 0$.

Notes a) We could write $f'(x) = 2|x|$ and we saw in the notes that $|x|$ is not differentiable at $x = 0$.

b) Given $n \geq 1$ could you construct a function that has n derivatives at 0, i.e. $f^{(i)}(0)$ exist for all $1 \leq i \leq n$, yet has no $n+1$ derivative at 0, i.e. $f^{(n+1)}(0)$ does not exist? **End of Notes**